ME 747 Introduction to computational fluid dynamics

Lecture 3

Overviews of governing equations for fluid flow and heat transfer

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Lecture schedule

Session	Topics
1	**1.Overviews of computational fluid dynamics** **-** Overviews and importance of heat transfer in real applications
2 - 3	**2. Introduction to Fortran programming** - Basic commands in Fortran programming
4	**3. Overviews of governing equations for flow and heat transfer** -Elliptic, Parabolic and Hyperbolic equations
5	**4. Introduction to numerical methods** - Finite different method, Finite volume method, Finite element method, etc.
6 – 7	**5. Introduction to solve engineering problems with finite-different method** - Taylor series expansion, Approximation of the second derivative, Initial condition and Boundary conditions

Contents

- Ordinary differential equations (ODEs)
- Partial differential equations (PDEs)

Ordinary differential equations (ODEs)

- **Classification for ordinary differential equations**

First order	Higher order
Single equations	System equation
Linear	Nonlinear
Initial value	Boundary value

สมการอันดับ (*n* th order equation)

$$y^n = f\left(x, y, y', \ldots, y^{n-1}\right)$$

สามารถลดอยู่ในระบบของสมการอันดับหนึ่ง โดยนิยาม

$$y_1 = y$$

$$y_j = y^{j-1} \qquad \text{ซึ่ง} \qquad j = 1, 2, \ldots, n-1$$

ดังนั้น

$$y'_j = y_{j+1} \qquad \text{ซึ่ง} \qquad j = 1, 2, \ldots, n-1$$

$$y'_n = f\left(x, y_1, y_2, \ldots, y_n\right)$$

Initial value problems

First-order ordinary differential equation

$$\frac{dy}{dt} = f(y,t) \qquad\qquad y(0) = y_0$$

เราต้องการหาค่า $y(t)$ สำหรับ $0 < t \le t_f$

สำหรับ numerical method ที่เวลา $t_{n+1} = t_n + \Delta t$ สำหรับ $0 \le t \le t_n$

ค่า y ณ เวลาถัดไปหาได้จาก $y_{n+1} = y(t_{n+1})$

Taylors series methods

หาคำตอบที่เวลา t_{n+1} รอบคำตอบที่เวลา t_n

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \ldots$$

ที่ซึ่ง $h = \Delta t$

จากสมการ $\frac{dy}{dt} = f(y,t)$ → $y'_n = f(y_n, t_n)$

ดังนั้นคำตอบสมการอันดับหนึ่ง

$$y_{n+1} = y_n + hf(y_n, t_n)$$

Equation forms

- Hyperbolic problems (waves):
  - Quantum mechanics: Wave-function(position,time)
- Elliptic (steady state) problems:
  - Electrostatic or Gravitational Potential: Potential(position)
- Parabolic (time-dependent) problems:
  - Heat flow: Temperature(position, time)
  - Diffusion: Concentration(position, time)

Many problems combine features of above

- Fluid flow: Velocity,Pressure,Density(position,time)
- Elasticity: Stress,Strain(position,time)

Classification of PDEs

- **Elliptic Type**
- **Parabolic Type**
- **Hyperbolic Type**
- different mathematical and physical behaviors

Fluid flow equations

- Time : first-derivative (second-derivative for wave eqn)
- Space: first- and second-derivatives
- System of coupled equations for several variables

Classification of PDEs

❑ **First-order linear wave equation (advection eq.)**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Propagation of wave with speed c

Advection of passive scalar with speed c

❑ **First-order nonlinear wave equation (inviscid Burgers's equation)**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Classification of PDEs

- **Advection-diffusion equation (linear)**

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

- **Burger's equation (nonlinear)**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$$

Other Common PDEs

- Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Nonlinear

dispersive wave

- Laplace and Poisson's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \begin{cases} f = 0 : \text{Laplace} \\ f \neq 0 : \text{Poisson} \end{cases}$$

Other Common PDEs

❑ **Helmholtz equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

Time-dependent harmonic waves

Propagation of acoustic waves

❑ **Tricomi equation**

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{cases} y > 0 & : \text{ elliptic} \\ y < 0 & : \text{ hyperbolic} \end{cases}$$

Mixed-type

Other Common PDEs

- Wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- Fourier equation (Heat equation)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Navier-Stokes Equations

- Navier-Stokes equation
- Primitive variables

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex] \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\[2ex] \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

Navier-Stokes Equations

- Navier-Stokes equation
- Vorticity / stream function formulation

$$\begin{cases} \nabla^2 \psi = -\omega \\ \dfrac{\partial \omega}{\partial t} + u \dfrac{\partial \omega}{\partial x} + v \dfrac{\partial \omega}{\partial y} = \nu \left(\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2} \right) \end{cases}$$

Parabolic PDEs

- One real (double) root, one characteristic direction (typically t = const)
- **The solution is marching in time (or spatially) with given initial conditions**
- The solution will be modified by the boundary conditions (time-dependent, in general) during the propagation
- Any change in boundary conditions at t_1 will not affect solution at $t < t_1$, but will change the solution after $t = t_1$

General 2nd order partial differential equations

- **Linear second-order PDE in two independent variables (x,y), (x,t), etc.**

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

A, B, C, …, G are constant coefficients (may be generalized)

$$\textbf{Classification} \quad \begin{cases} B^2 - 4AC < 0 : \textbf{elliptic} \\ B^2 - 4AC = 0 : \textbf{parabolic} \\ B^2 - 4AC > 0 : \textbf{hyperbolic} \end{cases}$$

Classification of PDEs

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu + G = 0$$

- The classification depends only on the highest-order derivatives (independent of D, E, F, G)
- For nonlinear problems [A,B,C = f(x,y,u)],
- Physical processes are independent of coordinates
- Introduction of simpler flow categories (approximations) may change the equation type

Steady-state : parabolic $\longrightarrow$ elliptic

Boundary-layer : elliptic $\longrightarrow$ parabolic

Classification of PDEs

❑ **General form of second-order PDEs (2 variables)**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu + G = 0$$

(1) <u>Hyperbolic PDEs</u> (Propagation)

Advection equation

Wave equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + u \dfrac{\partial \phi}{\partial x} = 0 & \text{(first - order)} \\ \dfrac{\partial^2 \phi}{\partial t^2} - c^2 \dfrac{\partial^2 \phi}{\partial x^2} = 0 & \text{(second} - \text{order)} \end{cases}$$

Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(2) <u>Parabolic PDEs</u> (Time- or space-marching)

Burger's equation

Fourier equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} = v\dfrac{\partial^2 \phi}{\partial x^2} \\ \dfrac{\partial \phi}{\partial t} = \alpha\dfrac{\partial^2 \phi}{\partial x^2} \end{cases}$$

Diffusion / dispersion

Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu + G = 0$$

(3) Elliptic PDEs (Diffusion, equilibrium problems)

Laplace equation

Possion's equation

Helmholtz equation

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0 \\ \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = f(x, y) \\ \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + c^2 \phi = 0 \end{cases}$$

Classification of PDEs

❑ **General form of second-order PDEs (2 variables)**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu + G = 0$$

(4) Mixed-type PDEs

Steady, compressible potential flow

$$(1 - M^2)\frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial n^2} = 0 \qquad \begin{cases} M < 1: & \text{subsonic} \\ M > 1: & \text{supersonic} \end{cases}$$

Classification of PDEs

❑ **General form of second-order PDEs (2 variables)**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(5) System of Coupled PDEs

Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

Equilibrium Problems

- **Boundary Value Problems**
- **Elliptic PDE**
- **“Jury”** Problem (every juror must agree on the same verdict)
- The entire solution is passed on a jury requiring satisfaction of all boundary conditions and all internal requirements
- Usually **“steady-state”**

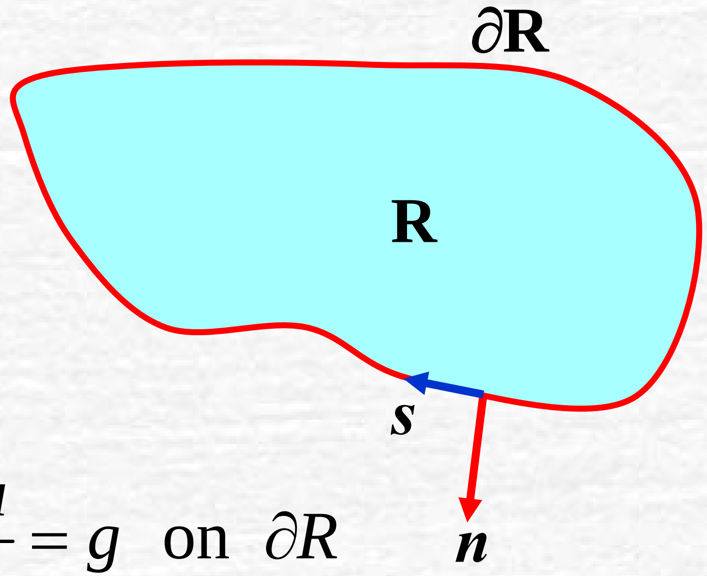
Propagation Problems

- **Initial Value Problems**
- **Hyperbolic or Parabolic**
- **“Marching” problems**
- Unsteady, transient, steady shock, boundary-layer (space-marching), …
- The solution marched out from the initial state guided and modified in transient by the side boundary conditions
- **Parabolic** – marching in certain direction, equilibrium in the other directions

Boundary and Initial Conditions

Initial conditions: starting point for propagation problems

Boundary conditions: specified on domain boundaries to provide the interior solution in computational domain



(i) Dirichlet condition: $u = f \quad \text{on} \quad \partial R$

(ii) Neumann condition: $\dfrac{\partial u}{\partial n} = f \;\text{ or }\; \dfrac{\partial u}{\partial s} = g \quad \text{on} \quad \partial R$

Second-Order PDEs

Second-order PDE in two variables

$$Au_{xx} + Bu_{xy} + Cu_{yy} + H = 0$$

$$let \begin{cases} P = u_x \\ Q = u_y \end{cases} \quad then \begin{cases} P_x = u_{xx}(= R),\ Q_y = u_{yy}(= T) \\ P_y = Q_x = u_{xy}(= S) \end{cases}$$

Express every derivative in terms of *uxy*

$$\begin{cases} dP = P_x dx + P_y dy = u_{xx} dx + u_{xy} dy \\ dQ = Q_x dx + Q_y dy = u_{xy} dx + u_{yy} dy \end{cases}$$

$$\Rightarrow u_{xx} = \frac{dP}{dx} - u_{xy} \frac{dy}{dx}; \quad u_{yy} = \frac{dQ}{dy} - u_{xy} \frac{dx}{dy}$$

Second-Order PDEs

$$Au_{xx} + Bu_{xy} + Cu_{yy} + H$$

$$= A\left(\frac{dP}{dx}\right) + C\left(\frac{dQ}{dy}\right) + H + u_{xy}\left[-A\left(\frac{dy}{dx}\right) + B - C\frac{dx}{dy}\right] = 0$$

- **Eliminate the dependence on partial derivatives**

Choose $-A\left(\frac{dy}{dx}\right) + B - C\frac{dx}{dy} = 0 \Leftrightarrow A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$

then $A\left(\frac{dP}{dx}\right) + C\left(\frac{dQ}{dy}\right) + H = 0$ **involves only total differentials**

Characteristic Equation

Characteristic equation for second-order PDE

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \;\Rightarrow\; \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

- **Classification of second-order PDEs**

$$\begin{cases} \text{Hyperbolic:} & B^2 - 4AC > 0, \text{ two real roots (characteristics)} \\ \text{Parabolic:} & B^2 - 4AC = 0, \text{ one real root (characteristics)} \\ \text{Elliptic:} & B^2 - 4AC < 0, \text{ two complex roots (cannot identify} \\ & \qquad \text{the propagation directions)} \end{cases}$$

Hyperbolic PDEs

- Two real roots, two characteristic directions
- **Two propagation (marching) directions**
- **Domain of dependence**
- **Domain of influence**
- (u_x,u_y,v_x,v_y) are not uniquely defined along the characteristic lines, discontinuity may occur
- Boundary conditions must be specified according to the characteristics

Elliptic PDEs

- The derivatives (u_x, u_y, v_x, v_y) can always be uniquely determined at every point in the solution domain
- **No marching or propagation direction !**
- Boundary conditions needed on all boundaries
- The solution will be continuous (smooth) in the entire solution domain
- **Jury problem** - all boundary conditions must be satisfied simultaneously