ME 747 Introduction to computational fluid dynamics

Lecture 5

Introduction to solve engineering problems with finite-difference method

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Lecture schedule

Session	Topics
1	**1.Overviews of computational fluid dynamics** - Overviews and importance of heat transfer in real applications
2 - 3	**2. Introduction to Fortran programming** - Basic commands in Fortran programming
4	**3. Overviews of governing equations for flow and heat transfer** -Elliptic, Parabolic and Hyperbolic equations
5	**4. Introduction to numerical methods** - Finite different method, Finite volume method, Finite element method, etc.
6 – 7	**5. Introduction to solve engineering problems with finite-difference method** - Taylor series expansion, Approximation of the second derivative, Initial condition and Boundary conditions

Contents

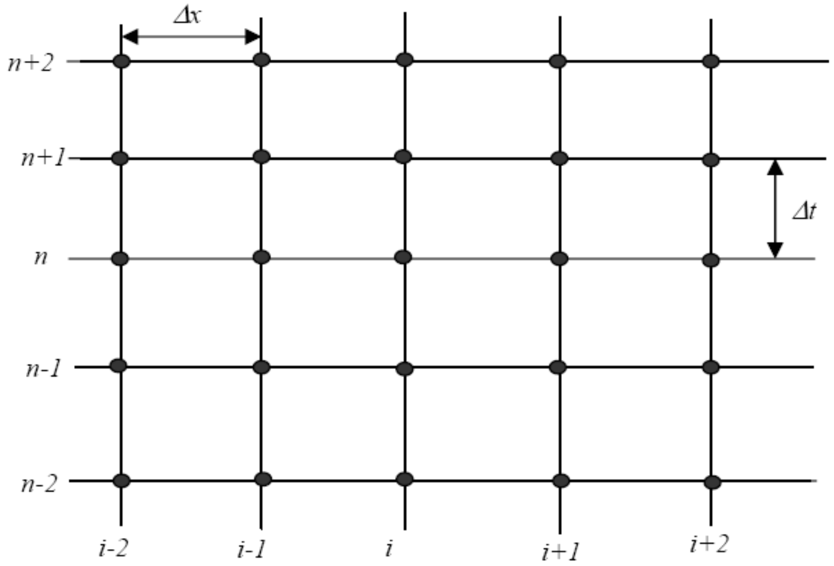
- Finite difference method
- Numerical discretization schemes

Solving the fluid dynamic equations

Transient –diffusion term

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

กำหนดให้ $\nu = 1$



❑ Explicit Euler method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} \qquad (2)$$

จัดรูปสมการได้เป็น

$$u_i^{n+1} = \frac{\Delta t}{\Delta x^2} u_{i-1}^n + \left[1 - 2\frac{\Delta t}{\Delta x^2}\right] u_i^n + \frac{\Delta t}{\Delta x^2} u_{i+1}^n \qquad (3)$$

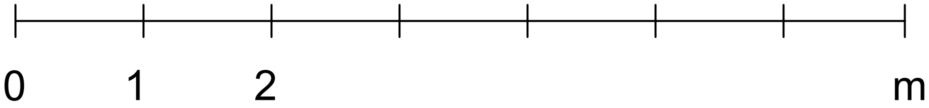
Explicit Euler method

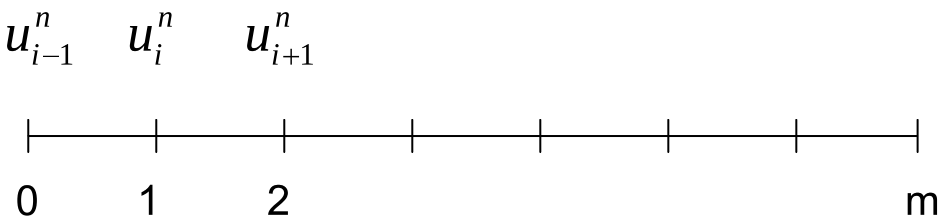
$$u_i^{n+1} = \frac{\Delta t}{\Delta x^2} u_{i-1}^n + \left[1 - 2\frac{\Delta t}{\Delta x^2}\right] u_i^n + \frac{\Delta t}{\Delta x^2} u_{i+1}^n$$

กำหนดให้ $u(0,x) = 5$

$u(t,0) = 1$

$u(t,L) = 1$





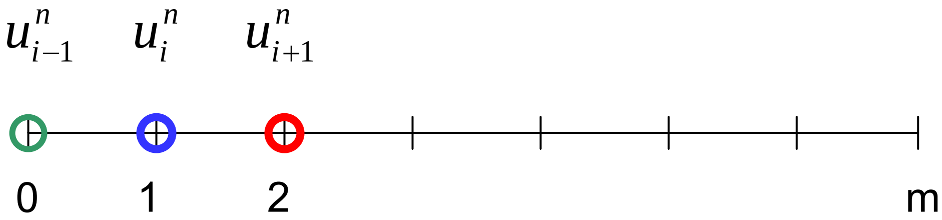
$$u_i^{n+1} = a u_{i-1}^n + b u_i^n + c u_{i+1}^n$$

โดยที่

$$a = \frac{\Delta t}{\Delta x^2} \qquad b = \left[1 - 2\frac{\Delta t}{\Delta x^2}\right] \qquad c = \frac{\Delta t}{\Delta x^2}$$

Explicit Euler method

$$u_i^{n+1} = au_{i-1}^n + bu_i^n + cu_{i+1}^n$$



Initial condition $u(0,x) = 5$ Boundary condition $u(t,0) = 1$ $u(t,L) = 1$

กำหนดให้ Δt = 0.5 และ Δx = 1

ดังนั้น a = 0.5, b = 0, c = 0.5

$$[a]\{u^n\} = \{u^{n+1}\} \rightarrow \begin{bmatrix} a_0 & b_0 & c_0 & & 0 \\ & a_1 & b_1 & c_1 & \\ & & a_2 & b_2 & c_2 \\ & & & a_3 & b_3 \\ 0 & & & & a_4 \end{bmatrix} \begin{Bmatrix} u_0^n \\ u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \end{Bmatrix} = \begin{Bmatrix} u_0^{n+1} \\ u_1^{n+2} \\ u_2^{n+3} \\ u_3^{n+4} \\ u_4^{n+5} \end{Bmatrix}$$

Explicit Euler method

At t = 0

$$u_0^0 = 1$$

$$u_1^0 = 5$$

$$u_2^0 = 5$$

$$u_i^{n+1} = au_{i-1}^n + bu_i^n + cu_{i+1}^n$$

At t = t + Δt

$$u_0^1 = 1$$

$$u_1^1 = (0.5)(1) + (0)(5) + (0.5)(5) = 3$$

$$u_2^1 = (0.5)(5) + (0)(5) + (0.5)(5) = 5$$

$$u_3^1 = (0.5)(5) + (0)(5) + (0.5)(5) = 5$$

$$u_4^1 = (0.5)(5) + (0)(5) + (0.5)(5) = 5$$

Explicit Euler method

At t = t + 2Δt

$$u_0^2 = 1$$

$$u_1^2 = (0.5)(1) + (0)(3) + (0.5)(5) = 3$$

$$u_2^2 = (0.5)(3) + (0)(5) + (0.5)(5) = 4$$

$$u_3^2 = (0.5)(5) + (0)(5) + (0.5)(5) = 5$$

At t = t + 3Δt

$$u_0^3 = 1$$

$$u_1^3 = (0.5)(1) + (0)(3) + (0.5)(4) = 2.5$$

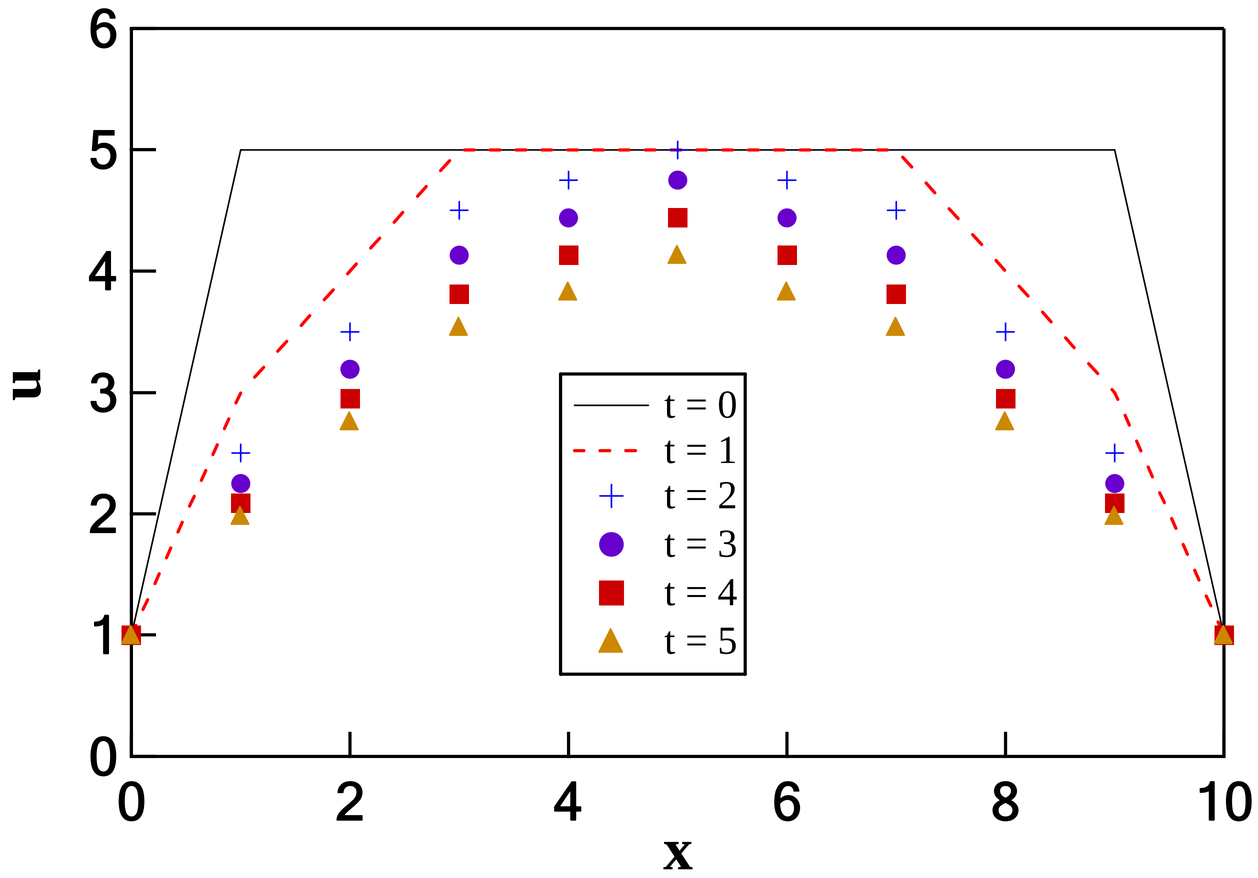
$$u_2^1 = (0.5)(3) + (0)(4) + (0.5)(5) = 4$$

$$u_3^1 = (0.5)(4) + (0)(5) + (0.5)(5) = 4.5$$

$$u_4^1 = (0.5)(5) + (0)(5) + (0.5)(5) = 5$$

Explicit Euler method

x	t = 0	1	2	3	4	5	6	7	8	9	10
.000E+00	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01
.100E+01	.500E+01	.300E+01	.300E+01	.250E+01	.250E+01	.225E+01	.225E+01	.209E+01	.209E+01	.198E+01	.198E+01
.200E+01	.500E+01	.500E+01	.400E+01	.400E+01	.350E+01	.350E+01	.319E+01	.319E+01	.295E+01	.295E+01	.276E+01
.300E+01	.500E+01	.500E+01	.500E+01	.450E+01	.450E+01	.413E+01	.413E+01	.381E+01	.381E+01	.354E+01	.354E+01
.400E+01	.500E+01	.500E+01	.500E+01	.500E+01	.475E+01	.475E+01	.444E+01	.444E+01	.413E+01	.413E+01	.383E+01
.500E+01	.500E+01	.500E+01	.500E+01	.500E+01	.500E+01	.475E+01	.475E+01	.444E+01	.444E+01	.413E+01	.413E+01
.600E+01	.500E+01	.500E+01	.500E+01	.500E+01	.475E+01	.475E+01	.444E+01	.444E+01	.413E+01	.413E+01	.383E+01
.700E+01	.500E+01	.500E+01	.500E+01	.450E+01	.450E+01	.413E+01	.413E+01	.381E+01	.381E+01	.354E+01	.354E+01
.800E+01	.500E+01	.500E+01	.400E+01	.400E+01	.350E+01	.350E+01	.319E+01	.319E+01	.295E+01	.295E+01	.276E+01
.900E+01	.500E+01	.300E+01	.300E+01	.250E+01	.250E+01	.225E+01	.225E+01	.209E+01	.209E+01	.198E+01	.198E+01
.100E+02	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01	.100E+01



Initial condition $u(0,x)=5$ $\quad\dfrac{\Delta t}{\Delta x^2}=\dfrac{0.5}{(1)^2}=0.5$

Boundary condition $u(t,0)=1$

$u(t,L)=1$

```fortran
      program diffusion
      parameter (n= 10,m=10)
      real u(0:n+1,0:m+1)
      open(1000,file='OUT-diffusion',status='unknown')
      dt = 0.5
      RX = 10.0
      dx = RX/real(m)
      dtdx = dt/dx/dx
      do 10 i = 0,m
      u(0,i) = 5.0
      u(0,0) = 1.0
      u(0,m) = 1.0
10    continue
      do 110 j = 1,n
      do 100 i = 1,m-1
      u(j,0) = 1.0
      u(j,m) = 1.0
      u(j,i) = dtdx*u(j-1,i-1)+(1.0-2.0*dtdx)*u(j-1,i)+dtdx*u(j-1,i+1)
100   continue
110   continue
      do 210 i = 0,m
      write(1000,1001) real(i)*dx,(u(j,i),j = 0,n)
1001  format(e8.3,x,11e8.3)
210   continue
      stop
      end
```

Assignment 2

- จงใช้วิธี Explicit Euler method คำนวณและวาดกราฟการเปลี่ยนแปลง ค่าความเร็ว u ณ ตำแหน่งและเวลาต่างๆ เมื่อ Δt = 0.05 และ Δx = 0.1, 0.2, 0.4, 1, 2

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

กำหนดให้ $\nu = 1$

Initial condition $u(0,x) = 5$

Boundary condition $u(t,0) = 0$

$u(t,L) = 0$

Implicit Euler method

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

กำหนดให้ $\nu = 1$

$$u(0,x) = 5$$

$$u(t,0) = 1$$

$$u(t,L) = 1$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}$$

$$u_i^{n+1} - u_i^n = \frac{\Delta t}{\Delta x^2} u_{i-1}^{n+1} - 2\frac{\Delta t}{\Delta x^2} u_i^{n+1} + \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1}$$

$$u_i^n = -\frac{\Delta t}{\Delta x^2} u_{i-1}^{n+1} + u_i^{n+1} + 2\frac{\Delta t}{\Delta x^2} u_i^{n+1} - \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1}$$

$$u_i^n = -\frac{\Delta t}{\Delta x^2} u_{i-1}^{n+1} + \left(1 + 2\frac{\Delta t}{\Delta x^2}\right) u_i^{n+1} - \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1}$$

Implicit Euler method

$$u_i^n = -\frac{\Delta t}{\Delta x^2} u_{i-1}^{n+1} + \left(1 + 2\frac{\Delta t}{\Delta x^2}\right) u_i^{n+1} - \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1}$$

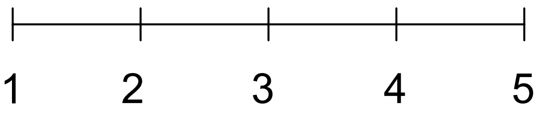
$$u_i^n = a u_{i-1}^{n+1} + b u_i^{n+1} + c u_{i+1}^{n+1}$$

$$a = -\frac{\Delta t}{\Delta x^2} \qquad b = \left(1 + 2\frac{\Delta t}{\Delta x^2}\right) \qquad c = -\frac{\Delta t}{\Delta x^2}$$

To solve this problem, need initial and boundary conditions

Implicit Euler method

สมมติว่าพิจารณา 5 grid



$$u_1^0 = +a_1u_1^1 + b_1u_2^1 + c_1u_3^1$$

$$u_2^0 = a_2u_2^1 + b_2u_3^1 + c_2u_4^1$$

$$u_3^0 = a_3u_3^1 + b_3u_4^1 + c_3u_5^1$$

$$u_4^0 = a_4u_4^1 + b_4u_5^1$$

$$u_5^0 = a_5u_5^1$$

$$\begin{bmatrix} a_1 & b_1 & c_1 & & \\ & a_2 & b_2 & c_2 & \\ & & a_3 & b_3 & c_3 \\ & & & a_4 & b_4 \\ & & & & a_5 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \end{Bmatrix} = \begin{Bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \\ u_4^0 \\ u_5^0 \end{Bmatrix}$$

$$[A]\{u_i^{n+1}\} = \{u_i^n\}$$

เมื่อตัดค่าที่ boundary conditions

$$\begin{bmatrix} a_2 & b_2 & c_2 \\ & a_3 & b_3 \\ & & a_4 \end{bmatrix} \begin{Bmatrix} u_2^1 \\ u_3^1 \\ u_4^1 \end{Bmatrix} = \begin{Bmatrix} u_2^0 \\ u_3^0 \\ u_4^0 \end{Bmatrix}$$

Implicit Euler method

$$\begin{bmatrix} a_1 & b_1 & c_1 & & & & & & & \\ & a_2 & b_2 & c_2 & & & & & & \\ & & a_3 & b_3 & c_3 & & & & & \\ & & & a_4 & b_4 & c_4 & & & & \\ & & & & a_5 & b_5 & c_5 & & & \\ & & & & & a_6 & b_6 & c_6 & & \\ & & & & & & a_7 & b_7 & c_7 & \\ & & & & & & & a_8 & b_8 & c_8 \\ & & & & & & & & a_9 & b_9 \\ & & & & & & & & & a_{10} \end{bmatrix} \begin{Bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1} \\ u_{10}^{n+1} \end{Bmatrix} = \begin{Bmatrix} u_1^{n} \\ u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ u_5^{n} \\ u_6^{n} \\ u_7^{n} \\ u_8^{n} \\ u_9^{n} \\ u_{10}^{n} \end{Bmatrix}$$

Implicit Euler method

Know initial and boundary conditions $u(0,x) = 5$ $u_1 = 1$ $u_{10} = 1$

$$\begin{bmatrix} a_2 & b_2 & c_2 & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & a_4 & b_4 & c_4 & & & \\ & & & a_5 & b_5 & c_5 & & \\ & & & & a_6 & b_6 & c_6 & \\ & & & & & a_7 & b_7 & c_7 \\ & & & & & & a_8 & b_8 \\ & & & & & & & a_9 \end{bmatrix} \begin{Bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1} \end{Bmatrix} = \begin{Bmatrix} u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ u_5^{n} \\ u_6^{n} \\ u_7^{n} \\ u_8^{n} \\ u_9^{n} \end{Bmatrix}$$

tridiagonal matrix algorithm (TDMA)

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

เมื่อ $a_1 = c_n = 0$

$$\begin{bmatrix} b_1 & c_1 & & & & & & 0 \\ a_2 & b_2 & c_2 & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & a_4 & b_4 & c_4 & & & \\ & & & a_5 & b_5 & c_5 & & \\ & & & & a_6 & b_6 & c_6 & \\ & & & & & a_7 & b_7 & c_7 \\ 0 & & & & & & a_8 & b_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ \\ x_8 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \\ \\ \\ \\ d_8 \end{bmatrix}$$

$$[A]\{x_i\} = \{d_i\}$$

Concept

ใช้ Matrix operation เพื่อทำให้สัมประสิทธิ์แถวทแยงมุม มีค่าเป็น 1

tridiagonal matrix algorithm (TDMA)

ขั้นที่ 1 หารเมตริกซ์แถวแรกด้วย b_1

$$
\begin{bmatrix}
1 & \frac{c_1}{b_1} & & & & & & 0 \\
a_2 & b_2 & c_2 & & & & & \\
 & a_3 & b_3 & c_3 & & & & \\
 & & a_4 & b_4 & c_4 & & & \\
 & & & a_5 & b_5 & c_5 & & \\
 & & & & a_6 & b_6 & c_6 & \\
 & & & & & a_7 & b_7 & c_7 \\
0 & & & & & & a_8 & b_8
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ x_8 \end{bmatrix}
=
\begin{bmatrix} \frac{d_1}{b_1} \\ d_2 \\ d_3 \\ \\ \\ \\ \\ d_8 \end{bmatrix}
$$

tridiagonal matrix algorithm (TDMA)

ขั้นที่ 2 คูณแถวที่หนึ่งด้วย a_2 แล้วนำไปลบออกจากแถวที่สองและกำหนดให้

$$c_1' = \frac{c_1}{b_1} \qquad \text{และ} \qquad d_1' = \frac{d_1}{b_1}$$

$$\begin{bmatrix} 1 & c_1' & & & & & & 0 \\ 0 & b_2 - c_1'a_2 & c_2 & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & a_4 & b_4 & c_4 & & & \\ & & & a_5 & b_5 & c_5 & & \\ & & & & a_6 & b_6 & c_6 & \\ & & & & & a_7 & b_7 & c_7 \\ 0 & & & & & & a_8 & b_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ \\ x_8 \end{bmatrix} = \begin{bmatrix} d_1' \\ d_2 - d_1'a_2 \\ d_3 \\ \\ \\ \\ \\ d_8 \end{bmatrix}$$

ขั้นที่ 3 หารแถวที่สองด้วย $b_2 - c_1' a_2$

$$\begin{bmatrix} 1 & c_1' & & & & & & 0 \\ 0 & 1 & \dfrac{c_2}{b_2 - c_1' a_2} & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & a_4 & b_4 & c_4 & & & \\ & & & a_5 & b_5 & c_5 & & \\ & & & & a_6 & b_6 & c_6 & \\ & & & & & a_7 & b_7 & c_7 \\ 0 & & & & & & a_8 & b_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ \\ x_8 \end{bmatrix} = \begin{bmatrix} d_1' \\ \dfrac{d_2 - d_1' a_2}{b_2 - c_1' a_2} \\ d_3 \\ \\ \\ \\ \\ d_8 \end{bmatrix}$$

$$\frac{d_2 - d_1' a_2}{b_2 - c_1' a_2} \rightarrow d_2'$$

และเริ่มต้นทำซ้ำขั้นที่สอง กับแถวอื่นๆ จนถึงแถวที่ n หรือเขียนได้เป็น

$$c_i' = \begin{cases} \dfrac{c_1}{b_1} & ; i = 1 \\ \dfrac{c_i}{b_i - c_{i-1}' a_i} & ; i = 2, 3, \ldots, n \end{cases} \qquad d_i' = \begin{cases} \dfrac{d_1}{b_1} & ; i = 1 \\ \dfrac{d_i - d_{i-1}' a_i}{b_i - c_{i-1}' a_i} & ; i = 2, 3, \ldots, n \end{cases}$$

$$
\begin{bmatrix}
1 & c_1' & & & & & & 0 \\
0 & 1 & \dfrac{c_2}{b_2 - c_1' a_2} & & & & & \\
 & 0 & 1 & \dfrac{c_3}{b_3 - c_2' a_3} & & & & \\
 & & 0 & 1 & \dfrac{c_4}{b_4 - c_3' a_4} & & & \\
 & & & 0 & 1 & \dfrac{c_5}{b_5 - c_4' a_5} & & \\
 & & & & 0 & 1 & \dfrac{c_6}{b_6 - c_5' a_6} & \\
 & & & & & 0 & 1 & \dfrac{c_7}{b_7 - c_6' a_7} \\
 & & & & & & 0 & 1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \end{bmatrix}
=
\begin{bmatrix} d_1' \\ \dfrac{d_2 - d_1' a_2}{b_2 - c_1' a_2} \\ \vdots \\ \dfrac{d_7 - d_6' a_7}{b_7 - c_6' a_7} \\ \dfrac{d_8 - d_7' a_8}{b_8 - c_7' a_8} \end{bmatrix}
$$

$$= d_n'$$

เมื่อค่าสัมประสิทธิ์ b_n = 1 แล้ว จะได้

$$x_n = d_n'$$

ขั้นสุดท้ายทำการแทนค่า x_n เพื่อหาค่า x_{n-1}, x_{n-2}, …., x_1

$$x_i = d_i' - c_i' x_{x+1} \quad ; i = n-1, n-2, \ldots, 1$$

```
      SUBROUTINE TRDIAG (N,A,B,C,X,Q)
      DIMENSION A(1000),B(1000),C(1000),X(1000),Q(1000),BB(1000)
C.....THIS SUBROUTINE SOLVES TRIDIAGONAL SYSTEMS OF EQUATIONS
C.....BY GAUSS ELIMINATION
C.....THE PROBLEM SOLVED IS MX=Q WHERE M=TRI(A,B,C)
C.....THIS ROUTINE DOES NOT DESTROY THE ORIGINAL MATRIX
C.....AND MAY BE CALLED A NUMBER OF TIMES WITHOUT REDEFINING
C.....THE MATRIX
C.....N = NUMBER OF EQUATIONS SOLVED (UP TO 1000)
C.....FORWARD ELIMINATION
C.....BB IS A SCRATCH ARRAY NEEDED TO AVOID DESTROYING B ARRAY
      DO 1 I=1,N
      BB(I) = B(I)
    1 CONTINUE
      DO 2 I=2,N
      T = A(I)/BB(I-1)
      BB(I) = BB(I) - C(I-1)*T
      Q(I) = Q(I) - Q(I-1)*T
    2 CONTINUE
C.....BACK SUBSTITUTION
      X(N) = Q(N)/BB(N)
      DO 3 I=1,N-1
      J = N-I
      X(J) = (Q(J)-C(J)*X(J+1))/BB(J)
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DTRIDG (N,A,B,C,X,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(1000),B(1000),C(1000),X(1000),G(1000),BB(1000)
C.....THIS SUBROUTINE SOLVES TRIDIAGONAL SYSTEMS OF EQUATIONS
C.....BY GAUSS ELIMINATION
C.....THE PROBLEM SOLVED IS MX=G WHERE M=TRI(A,B,C)
C.....THIS ROUTINE DOES NOT DESTROY THE ORIGINAL MATRIX
C.....AND MAY BE CALLED A NUMBER OF TIMES WITHOUT REDEFINING
C.....THE MATRIX
C.....N = NUMBER OF EQUATIONS SOLVED (UP TO 1000)
C.....FORWARD ELIMINATION
C.....BB IS A SCRATCH ARRAY NEEDED TO AVOID DESTROYING B ARRAY
      DO 1 I=1,N
      BB(I) = B(I)
    1 CONTINUE
      DO 2 I=2,N
      T = A(I)/BB(I-1)
      BB(I) = BB(I) - C(I-1)*T
      G(I) = G(I) - G(I-1)*T
    2 CONTINUE
C.....BACK SUBSTITUTION
      X(N) = G(N)/BB(N)
      DO 3 I=1,N-1
      J = N-I
      X(J) = (G(J)-C(J)*X(J+1))/BB(J)
    3 CONTINUE
      RETURN
      END
```

จากตัวอย่าง Implicit Euler method

$$\begin{bmatrix} a_2 & b_2 & c_2 & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & a_4 & b_4 & c_4 & & & \\ & & & a_5 & b_5 & c_5 & & \\ & & & & a_6 & b_6 & c_6 & \\ & & & & & a_7 & b_7 & c_7 \\ & & & & & & a_8 & b_8 \\ & & & & & & & a_9 \end{bmatrix} \begin{Bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1} \end{Bmatrix} = \begin{Bmatrix} u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ u_5^{n} \\ u_6^{n} \\ u_7^{n} \\ u_8^{n} \\ u_9^{n} \end{Bmatrix}$$

หารแถวแรกด้วย a_2

$$
\begin{bmatrix}
1 & \frac{b_2}{a_2} & \frac{c_2}{a_2} & & & & & \\
 & a_3 & b_3 & c_3 & & & & \\
 & & a_4 & b_4 & c_4 & & & \\
 & & & a_5 & b_5 & c_5 & & \\
 & & & & a_6 & b_6 & c_6 & \\
 & & & & & a_7 & b_7 & c_7 \\
 & & & & & & a_8 & b_8 \\
 & & & & & & & a_9
\end{bmatrix}
\begin{Bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1} \end{Bmatrix}
=
\begin{Bmatrix} \frac{u_2^n}{a_2} \\ u_3^n \\ u_4^n \\ u_5^n \\ u_6^n \\ u_7^n \\ u_8^n \\ u_9^n \end{Bmatrix}
$$

แล้วกำหนดให้

$$
\frac{b_2}{a_2} = b_2' \qquad \frac{c_2}{a_2} = c_2' \qquad \frac{u_2^n}{a_2} = u_2^{n'}
$$

นำแถวที่สองลบออกด้วยแถวที่หนึ่งและหารด้วย $a_3 - b_2'$

$$
\begin{bmatrix}
1 & b_2' & c_2' & & & & & & \\
 & 1 & \dfrac{b_3 - c_2' b_2'}{a_3 - b_2'} & \dfrac{c_3}{a_3 - b_2'} & & & & & \\
 & & a_4 & b_4 & c_4 & & & & \\
 & & & a_5 & b_5 & c_5 & & & \\
 & & & & a_6 & b_6 & c_6 & & \\
 & & & & & a_7 & b_7 & c_7 & \\
 & & & & & & a_8 & b_8 & \\
 & & & & & & & a_9 &
\end{bmatrix}
\begin{Bmatrix}
u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1}
\end{Bmatrix}
=
\begin{Bmatrix}
u_2^{n'} \\ \dfrac{u_3^n - u_2^{n'}}{a_3 - b_2'} \\ u_4^n \\ u_5^n \\ u_6^n \\ u_7^n \\ u_8^n \\ u_9^n
\end{Bmatrix}
$$

$$\frac{u_3^n - u_2^{n'}}{a_3 - b_2'} \rightarrow u_3^{n'}$$

ทำดังขั้นตอนที่ผ่านมาจนกระทั่งค่าสัมประสิทธิ์แถวทแยงมุมทุกตัวมีค่าเป็น 1

$$
\begin{bmatrix}
1 & b_2' & c_2' & & & & & \\
 & 1 & \dfrac{b_3-c_2'b_2'}{a_3-b_2'} & \dfrac{c_3}{a_3-b_2'} & & & & \\
 & & 1 & \dfrac{b_4-c_3'b_3'}{a_4-b_3'} & \dfrac{c_4}{a_4-b_3'} & & & \\
 & & & 1 & \dfrac{b_5-c_4'b_4'}{a_5-b_4'} & \dfrac{c_5}{a_5-b_4'} & & \\
 & & & & 1 & \dfrac{b_6-c_5'b_5'}{a_6-b_5'} & \dfrac{c_6}{a_6-b_5'} & \\
 & & & & & 1 & \dfrac{b_7-c_6'b_6'}{a_7-b_6'} & \dfrac{c_7}{a_7-b_6'} \\
 & & & & & & 1 & \dfrac{b_8-c_7'b_7'}{a_8-b_7'} \\
 & & & & & & & 1
\end{bmatrix}
\begin{Bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ u_6^{n+1} \\ u_7^{n+1} \\ u_8^{n+1} \\ u_9^{n+1} \end{Bmatrix}
=
\begin{Bmatrix} u_2^{n'} \\ \dfrac{u_3^n-u_2^{n'}}{a_3-b_2'} \\ \dfrac{u_4^n-u_3^{n'}}{a_4-b_3'} \\ \dfrac{u_5^n-u_4^{n'}}{a_5-b_4'} \\ \dfrac{u_6^n-u_5^{n'}}{a_6-b_5'} \\ \dfrac{u_7^n-u_6^{n'}}{a_7-b_6'} \\ \dfrac{u_8^n-u_7^{n'}}{a_8-b_7'} \\ \dfrac{u_9^n-u_8^{n'}}{a_9-b_8'} \end{Bmatrix}
$$

$$\frac{b_8-c_7'b_7'}{a_8-b_7'}$$

$$\frac{u_8^n-u_7^{n'}}{a_8-b_7'}$$

$$x_i = d_i' - c_i' x_{x+1} \quad ; i = n-1, n-2, \ldots, 1$$

จะได้ว่า $u_9^{n+1} = \dfrac{u_8^n - u_7^{n'}}{a_8-b_7'} = u_9^{n'}$ นำไปแทนค่า $u_8^{n+1} = u_9^{n'} - \dfrac{b_8-c_7'b_7'}{a_8-b_7'} u_8^{n'}$

Other methods

Richardson method

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=\left[\frac{u_{i-1}^{n}-2u_i^{n}+u_{i+1}^{n}}{\Delta x^2}\right]$$

DuFort-Frankel method

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=\left[\frac{u_{i-1}^{n}-u_i^{n+1}-u_i^{n-1}+u_{i+1}^{n}}{\Delta x^2}\right]$$

แทนเทอม diffusion ด้วย $\dfrac{u_i^{n+1}+u_i^{n-1}}{2}$

Truncation error $O\left[(\Delta t)^2,(\Delta x)^2,(\Delta t/\Delta x)^2\right]$

Other methods

Lassonen method (Euler implicit method)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \left[\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} \right]$$

Truncation error $O\left[\left(\Delta t\right), \left(\Delta x\right) \right]$

Crank-Nicolson method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2} \left[\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} \right] + \frac{1}{2} \left[\frac{u_{i-1}^{n} - 2u_i^{n} + u_{i+1}^{n}}{\Delta x^2} \right]$$

Assignment 3

- จงใช้วิธี Implicit Euler method คำนวณและวาดกราฟการเปลี่ยนแปลง ค่าความเร็ว u ณ ตำแหน่งและเวลาต่างๆ เมื่อ Δt = 0.05 และ Δx = 0.1, 0.2, 0.4, 1, 2

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

กำหนดให้ $\nu = 1$

Initial condition $u(0,x) = 5$

Boundary condition $u(t,0) = 0$

$u(t,L) = 0$